

GENERALIZED FREE TIME-DEPENDENT SCHRÖDINGER EQUATION WITH INITIAL DATA IN FOURIER LEBESGUE SPACES

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ABSTRACT. Consider the solution of the free time-dependent Schrödinger equation with initial data f . It is shown by Sjögren and Sjölin [5] that there exists f in the Sobolev space $H^s(\mathbf{R}^d)$, $s = d/2$ such that tangential convergence can not be widened to convergence regions. In [3] we obtain the corresponding results for a generalized version of the Schrödinger equation, where $-\Delta_x$ is replaced by an operator $\varphi(D)$, with special conditions on φ . In this paper we show that similar results may be obtained for initial data in Fourier Lebesgue spaces.

1. INTRODUCTION

In this paper we establish non-existence results of non-tangential convergence for the solution $u = S^\varphi f$ to the generalized time-dependent Schrödinger equation

$$(\varphi(D) + i\partial_t)u = 0, \quad (1.1)$$

with the initial condition

$$u(x, 0) = f(x).$$

Here φ is real-valued, and its radial derivatives of first and second orders ($\varphi' = \varphi'_r$ and $\varphi'' = \varphi''_{rr}$) are continuous outside a compact set containing origin, and fulfill appropriate growth conditions. In particular $\varphi(\xi) = |\xi|^a$ will satisfy these conditions, for $a > 1$. The exact conditions of admissible functions are given later on and we refer to [3] for further examples of admissible functions φ . By non-tangential convergence we mean convergence to initial data as time goes to zero and the space variable depends on the time non-linearly, i.e. the space

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variable is not fixed (as for convergence along vertical lines), nor linearly dependent of time (as for convergence along arbitrary straight lines). Furthermore, we consider initial datas in the weighted Fourier Lebesgue space, $\mathcal{FL}_s^p = \mathcal{FL}_{(\omega)}^p$, where $\omega(x, \xi) = \langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$, as well as for mixed weighted Fourier Lebesgue spaces,

$$\mathcal{FL}_{s_1, s_2}^{p, q}(\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}) = \mathcal{FL}_{s_1, s_2}^{p, q}(\mathbf{R}^d) = \mathcal{FL}_{(\omega)}^{p, q}(\mathbf{R}^d),$$

where $\omega(x, \xi) = \omega(x, \xi_1, \xi_2) = \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2}$ and $\xi_1 \in \mathbf{R}^{d_1}$, $\xi_2 \in \mathbf{R}^{d_2}$.

For $p = 2$ and initial data which belongs to $\mathcal{FL}_{d(p-1)/p}^p(\mathbf{R}^d)$ we recover Theorem 1.1 in [3]. There we proved existence of a function f in the Sobolev space $H^{d/2} = \mathcal{FL}_{d/2}^2$ such that near the vertical line $t \mapsto (x, t)$ through an arbitrary point $(x, 0)$ there are points accumulating at $(x, 0)$ such that the solution of equation (1.1) takes values far from f . This means that the solution of the time-dependent Schrödinger equation with initial condition $u(x, 0) = f(x)$ does not converge non-tangentially to f . Therefore we can not consider regions of convergence.

In this paper, we prove that the corresponding results hold for functions $f \in \mathcal{FL}_{d(p-1)/p}^p(\mathbf{R}^d)$ for $p \in (1, \infty]$. In the proof we use some ideas by Sjögren and Sjölin in [5] as well as [3], to construct a counter example. Some ideas can also be found in Sjölin [7, 8] and Walther [11, 12], and some related results are given in Bourgain [2], Kenig, Ponce and Vega [4], and Sjölin [6, 9]. The result in [3] is a special case of the result obtained here and the techniques here are similar.

Existence of regions of convergence has been studied before for other equations. For example, Stein and Weiss consider in [10, Chapter II Theorem 3.16] Poisson integrals acting on Lebesgue spaces. These operators are related to the operator S^φ .

For an appropriate function φ on \mathbf{R}^d , let S^φ be the operator acting on functions f defined by

$$f \mapsto \mathcal{F}^{-1}(\exp(it\varphi(\xi))\mathcal{F}f), \quad (1.2)$$

where $\mathcal{F}f$ is the Fourier transform of f , which takes the form

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) \equiv \int_{\mathbf{R}^d} e^{-ix \cdot \xi} f(x) dx,$$

when $f \in L^1(\mathbf{R}^d)$. This means that, if \widehat{f} is an integrable function, then S^φ in (1.2) takes the form

$$S^\varphi f(x, t) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{ix \cdot \xi} e^{it\varphi(\xi)} \widehat{f}(\xi) d\xi, \quad x \in \mathbf{R}^d, \quad t \in \mathbf{R}.$$

If $\varphi(\xi) = |\xi|^2$ and f belongs to the Schwartz class $\mathcal{S}(\mathbf{R}^d)$, then $S^\varphi f$ is the solution to the time-dependent Schrödinger equation $(-\Delta_x + i\partial_t)u = 0$ with the initial condition $u(x, 0) = f(x)$.

For more general appropriate φ , for which the equation (1.1) is well-defined, the expression $S^\varphi f$ is the solution to the generalized time-dependent Schrödinger equation (1.1) with the initial condition $u(x, 0) = f(x)$. Note here that $S^\varphi f$ is well-defined for any real-valued measurable φ and $f \in \mathcal{S}$. On the other hand, it might be difficult to interpret (1.1) if for example $\varphi \notin L^1_{loc}$.

In order to state the main result we need to specify the conditions on φ and give some definitions. The function φ should satisfy the conditions

$$\liminf_{r \rightarrow \infty} \left(\inf_{|\omega|=1} |\varphi'(r, \omega)| \right) = \infty, \quad (1.3)$$

and

$$\sup_{r \geq R} \left(\sup_{|\omega|=1} \frac{r^\beta |\varphi''(r, \omega)|}{|\varphi'(r, \omega)|^2} \right) < C, \quad (1.4)$$

for some $\beta > 0$ and some constant C . Here $\varphi'(r\omega) = \varphi'(r, \omega)$ denotes the derivative of $\varphi(r, \omega)$ with respect to r , and similarly for higher orders of derivatives.

In particular, $\varphi(\xi) = |\xi|^a$ is an appropriate function for $a > 1$ and $S^\varphi f(x, t)$ is then the solution to the generalized time-dependent Schrödinger equation $((-\Delta_x)^{a/2} + i\partial_t)u = 0$. For $a = 2$ this is the solution to the time-dependent Schrödinger equation $(-\Delta_x + i\partial_t)u = 0$ and this case is treated in Sjögren and Sjölin [5]. Some additional examples of appropriate functions φ can be found in [3].

Let $p \in [1, \infty]$ and $\omega \in \mathcal{P}(\mathbf{R}^d)$. The (weighted) Fourier Lebesgue space $\mathcal{FL}^p_{(\omega)}(\mathbf{R}^d)$ is the inverse Fourier image of $L^p_{(\omega)}(\mathbf{R}^d)$, i. e. $\mathcal{FL}^p_{(\omega)}(\mathbf{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{FL}^p_{(\omega)}} \equiv \|\widehat{f} \cdot \omega\|_{L^p}, \quad (1.5)$$

is finite. If $\omega = 1$, then the notation $\mathcal{F}L^p$ is used instead of $\mathcal{F}L_{(\omega)}^p$. We note that if $\omega(\xi) = \langle \xi \rangle^s$, then $\mathcal{F}L_{(\omega)}^p$ is the Fourier image of the Bessel potential space H_s^p (cf. [1]).

Here and in what follows we use the notation $\mathcal{F}L_s^p = \mathcal{F}L_{(\omega)}^p$ when $\omega(\xi) = \langle \xi \rangle^s$ so

$$\|f\|_{\mathcal{F}L_s^p(\mathbf{R}^d)}^p \equiv \int_{\mathbf{R}^d} \langle \xi \rangle^{sp} |\widehat{f}(\xi)|^p d\xi < \infty. \quad (1.6)$$

Theorem 1.1. *Assume that the function $\gamma : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is strictly increasing and continuous such that $\gamma(0) = 0$. Let $R > 0$, and let φ be real-valued functions on \mathbf{R}^d such that $\varphi'(r, \omega)$ and $\varphi''(r, \omega)$ are continuous and satisfy (1.3) and (1.4) when $r > R$. Also let $p \in (1, \infty)$. Then there exists a function $f \in \mathcal{F}L_s^p(\mathbf{R}^d)$, where $s = d(p-1)/p$, such that $S^\varphi f$ is continuous in $\{(x, t); t > 0\}$ and*

$$\limsup_{(y,t) \rightarrow (x,0)} |S^\varphi f(y, t)| = +\infty \quad (1.7)$$

for all $x \in \mathbf{R}^d$, where the limit superior is taken over those (y, t) for which $|y - x| < \gamma(t)$ and $t > 0$.

Furthermore, if $p = 1$ then the corresponding result holds for any $s < 0$ and if $p = \infty$ the result holds for $s = d$.

Here we recall that $\varphi' = \varphi'_r$ and $\varphi'' = \varphi''_{rr}$ are the first and second orders radial derivatives of φ . When $p \in (1, \infty)$ and $s > d(p-1)/p$, $p = 1$ and $s \geq 0$, or $p = \infty$ and $s > d$, no counter example of the form in Theorem 1.1 can be provided, since $S^\varphi f(y, t)$ converges to $f(x)$ as (y, t) approaches $(x, 0)$ non-tangentially when $f \in \mathcal{F}L_s^p(\mathbf{R}^d)$. In fact, Hölder's inequality gives

$$\begin{aligned} (2\pi)^d |S^\varphi f(x, t)| &\leq \int_{\mathbf{R}^d} |\widehat{f}(\xi)| d\xi \leq \|\langle \cdot \rangle^{-s}\|_{L^{p'}(\mathbf{R}^d)} \|\widehat{f} \langle \cdot \rangle^s\|_{L^p(\mathbf{R}^d)} \\ &\leq \|\langle \cdot \rangle^{-s}\|_{L^{p'}(\mathbf{R}^d)} \|f\|_{\mathcal{F}L_s^p(\mathbf{R}^d)}, \end{aligned}$$

which is finite when $f \in \mathcal{F}L_s^p(\mathbf{R}^d)$, $s > d(p-1)/p$ for some $p \in (1, \infty)$. Here p' is the conjugate exponent, i.e. $1/p + 1/p' = 1$. Therefore convergence along vertical lines can be extended to convergence regions when $s > d(p-1)/p$ and f belongs to $\mathcal{F}L_s^p(\mathbf{R}^d)$. We note that the estimates still hold for $p = 1$ and $p = \infty$, however it follows directly from the first inequality that $S^\varphi f(x, t)$ is finite for $f \in \mathcal{F}L_s^1(\mathbf{R}^d)$ and

$s \geq 0$. For $p = \infty$ we have that $S^\varphi f(x, t)$ is finite for $f \in \mathcal{FL}_s^\infty(\mathbf{R}^d)$ and $s > d$.

2. NOTATION FOR THE PROOFS

In order to prove Theorem 1.1 we introduce some notations. Let $B_r(x)$ be the open ball in \mathbf{R}^d with center at x and radius r . Numbers denoted by C , c or C' may be different at each occurrence. We let

$$\delta_k = \delta_{k,d} \equiv \gamma(1/(k+1))/\sqrt{d}, \quad k \in \mathbf{N},$$

where γ is the same as in Theorem 1.1. Since γ is strictly increasing it is clear that $(\delta_k)_{k \in \mathbf{N}}$ is strictly decreasing. We also let $(x_j)_{j=1}^\infty \subset \mathbf{R}^d$ be chosen such that x_1, x_2, \dots, x_{m_1} denotes all points in $B_1(0) \cap \delta_1 \mathbf{Z}^d$, $x_{m_1+1}, \dots, x_{m_2}$ denotes all points in $B_2(0) \cap \delta_2 \mathbf{Z}^d$ and generally

$$\{x_{m_k+1}, \dots, x_{m_{k+1}}\} = B_{k+1}(0) \cap \delta_{k+1} \mathbf{Z}^d, \quad \text{for } k \geq 1.$$

Furthermore we choose a strictly decreasing sequence $(t_j)_1^\infty$ such that $1 > t_1 > t_2 > \dots > 0$ and

$$\frac{1}{k+2} < t_j < \frac{1}{k+1}, \quad k \in \mathbf{N},$$

for $m_k + 1 \leq j \leq m_{k+1}$.

In the proof of Theorem 1.1 we consider the function f_φ , which is defined by the formula

$$\widehat{f}_\varphi(\xi) = \widehat{f}_{\varphi,B}(\xi) = |\xi|^{-d} (\log |\xi|)^{-B} \sum_{j=1}^\infty \chi_j(\xi) e^{-i(x_j \cdot \xi + t_j \varphi(\xi))}, \quad (2.1)$$

where for $p \in (1, \infty]$ fixed, we may fixate B such that $1/p < B < 1$. For $p = 1$ any B , such that $0 < B < 1$, suffices. We also have that χ_j is the characteristic function of

$$\Omega_j = \{\xi \in \mathbf{R}^d; R_j < |\xi| < R'_j\}.$$

Here $(R_j)_1^\infty$ and $(R'_j)_1^\infty$ are sequences in \mathbf{R} which fulfill the following conditions:

- (1) $R_1 \geq 2 + R$, $R'_1 \geq R_1 + 1$, with R given by Theorem 1.1;
- (2) $R'_j = R_j^N$ when $j \geq 2$, where N is a large positive number and independent of j , which is specified later on;

(3) $R_j < R'_j < R_{j+1}$, when $j \geq 1$;

(4)
$$|\varphi'(r, \omega)| > 1 \quad \text{when} \quad r \geq R; \quad (2.2)$$

(5) for $j \geq 2$

$$R_j^{\min(\beta, 1)} > \max_{l < j} \frac{2^j}{t_l - t_j}, \quad (2.3)$$

where $\beta > 0$ is the same constant as in (1.4) and

$$\inf_{R_j \leq r \leq R'_j} \left(\inf_{|\omega|=1} |\varphi'(r, \omega)| \right) > \max_{l < j} \frac{2|x_l - x_j|}{t_l - t_j}. \quad (2.4)$$

Remark 2.1. The sequences $(R_j)_1^\infty$ and $(R'_j)_1^\infty$ can be chosen since φ satisfies condition (1.3).

Furthermore, in order to get convenient approximations of the operator S^φ , we let

$$S_m^\varphi f(x, t) = \frac{1}{(2\pi)^d} \int_{|\xi| < R'_m} e^{ix \cdot \xi} e^{it\varphi(\xi)} \widehat{f}(\xi) d\xi. \quad (2.5)$$

Then

$$S_m^\varphi f_\varphi(x, t) = \sum_{j=1}^m A_j^\varphi(x, t), \quad (2.6)$$

where

$$A_j^\varphi(x, t) = \frac{1}{(2\pi)^d} \int_{\Omega_j} e^{i(x-x_j) \cdot \xi} e^{i(t-t_j)\varphi(\xi)} |\xi|^{-d} (\log |\xi|)^{-B} d\xi. \quad (2.7)$$

By using polar coordinates we get

$$A_j^\varphi(x_k, t_k) = \frac{1}{(2\pi)^d} \int_{|\omega|=1} \left\{ \int_{R_j}^{R'_j} \frac{1}{r(\log r)^B} e^{iF_\varphi(r, \omega)} dr \right\} d\sigma(\omega), \quad (2.8)$$

where

$$F_\varphi(r, \omega) = r(x_k - x_j) \cdot \omega + (t_k - t_j)\varphi(r, \omega),$$

and $d\sigma(\omega)$ is the euclidean surface measure on the $d - 1$ -dimensional unit sphere. By differentiation we get

$$F'_\varphi(r, \omega) = (x_k - x_j) \cdot \omega + (t_k - t_j)\varphi'(r, \omega) \quad (2.9)$$

and

$$F''_\varphi(r, \omega) = (t_k - t_j)\varphi''(r, \omega). \quad (2.10)$$

Here recall that $F'_\varphi(r\omega) = F'_\varphi(r, \omega)$ and $F''_\varphi(r, \omega)$ denote the first and second orders of derivatives of $F_\varphi(r, \omega)$ with respect to the r -variable.

By integration by parts in the inner integral of (2.8) we get

$$\int_{R_j}^{R'_j} \frac{1}{r(\log r)^B} e^{iF_\varphi(r, \omega)} dr = A_\varphi - B_\varphi, \quad (2.11)$$

where

$$A_\varphi = \left[\frac{e^{iF_\varphi(r, \omega)}}{r(\log r)^B i F'_\varphi(r, \omega)} \right]_{R_j}^{R'_j} \quad (2.12)$$

and

$$B_\varphi = \int_{R_j}^{R'_j} \frac{d}{dr} \left(\frac{1}{r(\log r)^B i F'_\varphi(r, \omega)} \right) e^{iF_\varphi(r, \omega)} dr. \quad (2.13)$$

3. PROOFS

To prove Theorem 1.1 we need some preparing lemmas. In the following lemma we prove that for fixed $x \in B_k(0)$ there exists sequences $(x_{n_j})_1^\infty$ and $(t_{n_j})_1^\infty$ such that

$$x_{n_j} \in \{x_{m_k+1}, \dots, x_{m_{k+1}}\}, \quad \text{and} \quad t_{n_j} \in \{t_{m_k+1}, \dots, t_{m_{k+1}}\}$$

and $|x_{n_j} - x| < \gamma(t_{n_j})$. The lemma is left without proof since the result can be found in [3].

Lemma 3.1. *Let $x \in \mathbf{R}^d$ be fixed. Then for each $k \geq |x|$ there exists $x_{n_j} \in \{x_{m_k+1}, \dots, x_{m_{k+1}}\}$ and $t_{n_j} \in \{t_{m_k+1}, \dots, t_{m_{k+1}}\}$ such that $|x_{n_j} - x| < \gamma(t_{n_j})$. In particular $(x_{n_j}, t_{n_j}) \rightarrow (x, 0)$ as j turns to infinity.*

We want to prove that f_φ in (2.1) belongs to $\mathcal{FL}_s^p(\mathbf{R}^d)$, with $s = d(p - 1)/p$, and fulfill (1.7). The former relation is a consequence of Lemma 3.2 below, which concerns Sobolev space properties for functions of the form

$$\widehat{g}(\xi) = |\xi|^{-d} (\log |\xi|)^{-\rho/p} \sum_{j=1}^{\infty} \chi_j(\xi) b_j(\xi), \quad (3.1)$$

where χ_j is the characteristic function on disjoint sets Ω_j .

Lemma 3.2. *Assume that $p \in (1, \infty)$, $\rho > 1$, Ω_j for $j \in \mathbf{N}$ are disjoint open subsets of $\mathbf{R}^d \setminus B_\mu(0)$ for some $\mu > 2$, $b_j \in L^1_{loc}(\mathbf{R}^d)$ for $j \in \mathbf{N}$ satisfies*

$$\sup_{j \in \mathbf{N}} \|b_j\|_{L^\infty(\Omega_j)} < \infty,$$

and let χ_j be the characteristic function for Ω_j . If g is given by (3.1), then $g \in \mathcal{F}L^p_s(\mathbf{R}^d)$, where $s = d(p-1)/p$.

For $p = 1$ and $\rho \geq 0$ it follows that $g \in \mathcal{F}L^1_s(\mathbf{R}^d)$ for any $s < 0$. Furthermore, if $p = \infty$ and ρ/p in (3.1) is replaced by $\rho \geq 0$, then $g \in \mathcal{F}L^\infty_s(\mathbf{R}^d)$ for $s = d$.

Proof. By estimating (1.6) for the function g when $p \in (1, \infty)$, $s = d(p-1)/p$ and the assumptions given by the lemma is satisfied, we get that

$$\begin{aligned} \int_{\mathbf{R}^d} |\widehat{g}(\xi)|^p \langle \xi \rangle^{sp} d\xi \\ \leq C \int_{\mathbf{R}^d \setminus B_\mu(0)} |\xi|^{-dp} (\log |\xi|)^{-\rho} \langle \xi \rangle^{sp} d\xi \\ \leq 2^{sp/2} C \int_\mu^\infty \frac{1}{r(\log r)^\rho} dr < \infty. \end{aligned}$$

The second inequality holds since $(1+r^2)^{sp/2} < (r^2+r^2)^{sp/2} = 2^{sp/2} r^{sp}$ for $r > 1$.

For $p = 1$ and $s < 0$, the second inequality, in the estimates above, is replaced by

$$\int_{\mathbf{R}^d \setminus B_\mu(0)} |\xi|^{-d} (\log |\xi|)^{-\rho} \langle \xi \rangle^s d\xi \leq 2^{s/2} C \int_\mu^\infty \frac{1}{r^{1-s} (\log r)^\rho} dr < \infty$$

For $p = \infty$ we have the norm

$$\begin{aligned} \text{ess sup}(|\widehat{g}(\xi)|^p \langle \xi \rangle^{sp}) &\leq C \text{ess sup}_{|\xi| > \mu} (|\xi|^{-d} (\log |\xi|)^{-\rho} \langle \xi \rangle^d) \\ &\leq 2^{d/2} (\log |\mu|)^{-\rho} < \infty. \end{aligned}$$

□

In the following lemma we give estimates of the expression A_j^φ .

Lemma 3.3. *Let $A_j^\varphi(x, t)$ be given by (2.7). Then the following is true:*

- (1) $\sum_{j=1}^{k-1} |A_j^\varphi(x, t)| \leq C(\log R'_{k-1})^{1-B}$, with C independent of k ;
- (2) $A_k^\varphi(x_k, t_k) > c(\log R'_k)^{1-B}$, with $c > 0$ independent of k .

Proof. (1) By triangle inequality and the fact that $|\xi| > 2$, when $\xi \in \Omega_j$, we get

$$\begin{aligned} \sum_{j=1}^{k-1} |A_j^\varphi(x, t)| &\leq \frac{1}{(2\pi)^d} \int_{2 \leq |\xi| \leq R'_{k-1}} |\xi|^{-d} (\log |\xi|)^{-B} d\xi \\ &= C \int_2^{R'_{k-1}} \frac{1}{r(\log r)^B} dr \leq C(\log R'_{k-1})^{1-B}, \end{aligned}$$

where C is independent of k . In the last equality we have taken polar coordinates as new variables of integration.

(2) Since $R_j^N = R'_j$ for sufficiently large N , we get

$$\begin{aligned} A_k^\varphi(x_k, t_k) &= C \int_{R_k}^{R'_k} \frac{1}{r(\log r)^B} dr \\ &= C \left((\log R'_k)^{1-B} - (\log(R'_k)^{1/N})^{1-B} \right) \\ &= C \left(1 - \frac{1}{N^{1-B}} \right) (\log R'_k)^{1-B} > c(\log R'_k)^{1-B}, \end{aligned}$$

for some constant $c > 0$, which is independent of k . \square

Lemma 3.4. *Assume that $S_m^\varphi f_\varphi$ is given by (2.5). Then $S_m^\varphi f_\varphi$ is continuous on $\{(x, t); t > 0, x \in \mathbf{R}^d\}$.*

Proof. The continuity for each $S_m^\varphi f_\varphi$ follows from the facts, that for almost every $\xi \in \mathbf{R}^d$, the map

$$(x, t) \mapsto e^{ix \cdot \xi} e^{it\varphi(\xi)} \widehat{f}_\varphi(\xi)$$

is continuous, and that

$$\int_{|\xi| < R'_m} |e^{ix \cdot \xi} e^{it\varphi(\xi)} \widehat{f}_\varphi(\xi)| d\xi = \int_{|\xi| < R'_m} |\widehat{f}_\varphi(\xi)| d\xi < C.$$

\square

When proving Theorem 1.1, we first prove that the modulus of $S_m^\varphi f_\varphi(x_k, t_k)$ turns to infinity as k goes to infinity. For this reason we note that the triangle inequality and (2.6) implies that

$$|S_m^\varphi f_\varphi(x_k, t_k)| \geq |A_k^\varphi(x_k, t_k)| - \left| \sum_{j=1}^{k-1} A_j^\varphi(x_k, t_k) \right| - \left| \sum_{j=k+1}^m A_j^\varphi(x_k, t_k) \right|, \quad (3.2)$$

where $m > k$. We want to estimate the terms in (3.2). From Lemma 3.3 we get estimates for the first two terms. It remains to estimate the last term.

Proof of Theorem 1.1.

Step 1. For $j > k \geq 2$ we shall estimate $|A_j^\varphi(x_k, t_k)|$ in (2.8). We have to find appropriate estimates for A_φ and B_φ in (2.11)-(2.13). By using $t_k - t_j > 0$ and $R_j < r < R'_j$ it follows from (2.4), (2.9), triangle inequality and Cauchy-Schwarz inequality that

$$\begin{aligned} |F'_\varphi(r, \omega)| &\geq (t_k - t_j)|\varphi'(r, \omega)| - |x_k - x_j| \\ &> (t_k - t_j)|\varphi'(r, \omega)| - (t_k - t_j) \frac{|\varphi'(r, \omega)|}{2} \\ &= \frac{|\varphi'(r, \omega)|}{2} (t_k - t_j). \end{aligned} \quad (3.3)$$

From (2.2), (2.3) and (3.3) it follows that

$$\begin{aligned} |A_\varphi| &= \left| \left[\frac{1}{r(\log r)^{Bi} F'_\varphi(r, \omega)} e^{iF_\varphi(r, \omega)} \right]_{R_j}^{R'_j} \right| \\ &\leq \frac{C}{R_j} \left(\frac{1}{|F'_\varphi(R_j, \omega)|} + \frac{1}{|F'_\varphi(R'_j, \omega)|} \right) \leq \frac{C}{(t_k - t_j)R_j} \leq C2^{-j}. \end{aligned}$$

In order to estimate B_φ , using (1.4), (2.10) and (3.3), we have

$$\begin{aligned} &\left| \frac{d}{dr} \left(\frac{1}{r(\log r)^{Bi} F'_\varphi(r, \omega)} \right) e^{iF_\varphi(r, \omega)} \right| \\ &\leq \frac{C}{r^2 |F'_\varphi(r, \omega)|} + \frac{C |F''_\varphi(r, \omega)|}{r |F'_\varphi(r, \omega)|^2 (\log r)^B} < \frac{C}{r^{1+\min(1, \beta)} (t_k - t_j)}. \end{aligned}$$

This together with (2.3) gives us

$$\begin{aligned} |B_\varphi| &= \left| \int_{R_j}^{R'_j} \frac{d}{dr} \left(\frac{1}{r(\log r)^B i F'_\varphi(r, \omega)} \right) e^{iF_\varphi(r, \omega)} dr \right| \\ &\leq \int_{R_j}^{R'_j} \frac{C}{r^{1+\min(1, \beta)}(t_k - t_j)} dr \leq \frac{C}{R_j^{\min(1, \beta)}(t_k - t_j)} \leq C2^{-j}. \end{aligned}$$

From the estimates above and the triangle inequality we get

$$|A_j^\varphi(x_k, t_k)| \leq C(|A_\varphi| + |B_\varphi|) < C2^{-j}, \quad j > k \geq 2. \quad (3.4)$$

Here C is independent of j and k .

Using the results from (3.2), (3.4), in combination with Lemma 3.3, and recalling that $R'_j = R_j^N$, gives us

$$\begin{aligned} |S_m^\varphi f_\varphi(x_k, t_k)| &\geq c(\log R'_k)^{1-B} - C'(\log R_k)^{1-B} - C \sum_{k+1}^m 2^{-j} \\ &\geq c(\log(R'_k))^{1-B} - \frac{C'}{N^{1-B}}(\log(R'_k))^{1-B} - C \geq c(\log R'_k)^{1-B}, \end{aligned} \quad (3.5)$$

when $m > k$ and N is chosen sufficiently large. Here $c > 0$ is independent of k .

Step 2. Now it remains to show that $S^\varphi f_\varphi$ is continuous when $t > 0$, and then it suffices to prove this continuity on a compact subset L of

$$\{(x, t); t > 0, x \in \mathbf{R}^d\}.$$

We want to replace (x_l, t_l) with $(x, t) \in L$ in (2.3) and (2.4). Since we have maximum over all l less than j , we can choose $j_0 < \infty$ large enough such that for all $j > l > j_0$ we have that $t_j < t_l < t$. Hence we may replace (x_l, t_l) with $(x, t) \in L$ on the right-hand sides in (2.3) and (2.4) for all $j > j_0$. This in turn implies that (3.4) holds when (x_k, t_k) is replaced by $(x, t) \in L$ and $j > j_0$. We use (3.4) to conclude that

$$\begin{aligned} &|S_m^\varphi f_\varphi(x, t) - S^\varphi f_\varphi(x, t)| \\ &= \left| (2\pi)^{-d} \int_{|\xi| < R'_m} e^{ix \cdot \xi} e^{it\varphi(\xi)} \widehat{f}_\varphi(\xi) d\xi - (2\pi)^{-d} \int_{\mathbf{R}^d} e^{ix \cdot \xi} e^{it\varphi(\xi)} \widehat{f}_\varphi(\xi) d\xi \right| \\ &= (2\pi)^{-d} \left| \int_{|\xi| > R'_m} e^{ix \cdot \xi} e^{it\varphi(\xi)} \widehat{f}_\varphi(\xi) d\xi \right| \leq C \sum_{i=m+1}^{\infty} 2^{-i} = C2^{-m}, \end{aligned}$$

when $m > j_0$. Hence $S_m^\varphi f_\varphi$ converge uniformly to $S^\varphi f_\varphi$ on every compact set.

We have now showed that $S_m^\varphi f_\varphi$ converge uniformly to $S^\varphi f_\varphi$ on every compact set and from Lemma 3.4 it follows that each $S_m^\varphi f_\varphi$ is a continuous function. Therefore it follows that $S^\varphi f_\varphi$ is continuous on $\{(x, t); t > 0\}$. In particular there is an $N \in \mathbf{N}$ such that

$$|S_m^\varphi f_\varphi(x_k, t_k) - S^\varphi f_\varphi(x_k, t_k)| < 1,$$

when $m > N$. Using (3.5) and the triangle inequality we get

$$\begin{aligned} c(\log R'_k)^{1-B} &\leq |S_m^\varphi f_\varphi(x_k, t_k)| \\ &\leq |S_m^\varphi f_\varphi(x_k, t_k) - S^\varphi f_\varphi(x_k, t_k)| + |S^\varphi f_\varphi(x_k, t_k)| < \\ &\qquad\qquad\qquad 1 + |S^\varphi f_\varphi(x_k, t_k)|. \end{aligned}$$

This gives us

$$|S^\varphi f_\varphi(x_k, t_k)| > c(\log R'_k)^{1-B} - 1 \rightarrow +\infty \text{ as } k \rightarrow +\infty.$$

For any fixed $x \in \mathbf{R}^d$ we can by Lemma 3.1 choose a subsequence (x_{n_j}, t_{n_j}) of (x_k, t_k) that goes to $(x, 0)$ as j turns to infinity. This gives the result. \square

4. MIXED FOURIER LEBESGUE SPACES

In this section we consider weighted mixed Fourier Lebesgue spaces as initial datas for the generalized free time-dependent Schrödinger equation. An analogous version of the previous results holds. Due to similarities in the proofs, as well as for the definition of initial data, we only need to show that the initial data belongs to the mixed Fourier Lebesgue space.

Let $p, q \in [1, \infty]$ and $d_1 + d_2 = d$, where $d_1, d_2 \in \mathbf{N}$. From now on we consider weights $\omega(\xi) = \omega(\xi_1, \xi_2)$, for $\xi_1 \in \mathbf{R}^{d_1}$ and $\xi_2 \in \mathbf{R}^{d_2}$, which are positive and $L^1_{\text{loc}}(\mathbf{R}^d)$. The (weighted) mixed Fourier Lebesgue space $\mathcal{FL}_{(\omega)}^{p,q}(\mathbf{R}^d) = \mathcal{FL}_{(\omega)}^{p,q}(\mathbf{R}^{d_1} \times \mathbf{R}^{d_2})$ consists of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\begin{aligned} \|f\|_{\mathcal{FL}_{(\omega)}^{p,q}(\mathbf{R}^d)} &\equiv \|\widehat{f} \cdot \omega\|_{L^{p,q}} \\ &\equiv \left(\int_{\mathbf{R}^{d_2}} \left(\int_{\mathbf{R}^{d_1}} |\widehat{f}(\xi_1, \xi_2)|^p \omega(\xi_1, \xi_2)^p d\xi_1 \right)^{q/p} d\xi_2 \right)^{1/q}, \end{aligned} \quad (4.1)$$

is finite. If $\omega = 1$, then the notation $\mathcal{FL}^{p,q}$ is used instead of $\mathcal{FL}_{(\omega)}^{p,q}$.

Here and in what follows we use the notation $\mathcal{F}L_{(\omega)}^{p,q} = \mathcal{F}L_{s_1,s_2}^{p,q}$ when $\omega(\xi_1, \xi_2) = \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2}$ so

$$\begin{aligned} & \|f\|_{\mathcal{F}L_{s_1,s_2}^{p,q}(\mathbf{R}^d)}^q \\ & \equiv \int_{\mathbf{R}^{d_2}} \langle \xi_2 \rangle^{s_2 q} \left(\int_{\mathbf{R}^{d_1}} \langle \xi_2 \rangle^{s_1 p} |\widehat{f}(\xi_1, \xi_2)|^p d\xi_1 \right)^{q/p} d\xi_2 < \infty. \end{aligned} \quad (4.2)$$

Theorem 4.1. *Assume that the function $\gamma : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is strictly increasing and continuous such that $\gamma(0) = 0$. Let $R > 0$, and let φ be real-valued functions on \mathbf{R}^d such that $\varphi'(r, \omega)$ and $\varphi''(r, \omega)$ are continuous and satisfy (1.3) and (1.4) when $r > R$. Also let $p, q \in (1, \infty)$ with $1/p + 1/q < 1$. Then there exists a function $f \in \mathcal{F}L_{s_1,s_2}^{p,q}(\mathbf{R}^d)$, where $s_1 = d_1(p-1)/p$ and $s_2 = d_2(q-1)/q$, such that $S^\varphi f$ is continuous in $\{(x, t); t > 0\}$ and*

$$\limsup_{(y,t) \rightarrow (x,0)} |S^\varphi f(y, t)| = +\infty \quad (4.3)$$

for all $x \in \mathbf{R}^d$, where the limit superior is taken over those (y, t) for which $|y - x| < \gamma(t)$ and $t > 0$. Note that for $p = \infty$ or $q = \infty$ the result holds when $s_1 = d_1$ or $s_2 = d_2$, respectively.

Remark 4.1. Note that $\mathcal{F}L^{p,p} = \mathcal{F}L^p$ holds independently of d_1 and d_2 . However the weights used in this section for mixed Lebesgue spaces, $\langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2}$ is not equivalent to those used for the L^p spaces, even when $p = q$. In fact, by choosing $p = q$ for these mixed spaces the weights are larger than the weights used for the usual Lebesgue spaces, and thereby the results in this section for the special case $p = q$ concern finding counter examples in a smaller space. It is here important to note that the result of previous section are not completely contained in the results obtained here, since the condition $1/p + 1/q < 1$ implies the requirement that $p > 2$ if $p = q$. No results for $p = q < 2$ are obtained in this section.

When $s_1 > d_1(p-1)/p$ and $s_2 > d_2(q-1)/q$ no counter example of the form in Theorem 4.1 can be provided, since $S^\varphi f(y, t)$ converges to $f(x)$ as (y, t) approaches $(x, 0)$ non-tangentially when $f \in \mathcal{F}L_{s_1,s_2}^{p,q}(\mathbf{R}^d)$.

In fact, Hölder's inequality gives

$$\begin{aligned}
(2\pi)^d |S^\varphi f(x, t)| &\leq \int_{\mathbf{R}^d} |\widehat{f}(\xi)| d\xi = \int_{\mathbf{R}^{d_2}} \int_{\mathbf{R}^{d_1}} |\widehat{f}(\xi_1, \xi_2)| d\xi_1 d\xi_2 \\
&\leq \|\langle \cdot \rangle^{-s_2}\|_{L^{q'}(\mathbf{R}^{d_2})} \left\| \int_{\mathbf{R}^{d_1}} |\widehat{f}(\xi_1, \cdot)| d\xi_1 \langle \cdot \rangle^{s_2} \right\|_{L^q(\mathbf{R}^{d_2})} \\
&\leq \|\langle \cdot \rangle^{-s_2}\|_{L^{q'}(\mathbf{R}^{d_2})} \|\langle \cdot \rangle^{-s_1}\|_{L^{p'}(\mathbf{R}^{d_1})} \|\widehat{f}(\cdot_1, \cdot_2) \langle \cdot_1 \rangle^{s_1} \langle \cdot_2 \rangle^{s_2}\|_{L^p(\mathbf{R}^{d_1})} \|_{L^q(\mathbf{R}^{d_2})} \\
&= \|\langle \cdot \rangle^{-s_2}\|_{L^{q'}(\mathbf{R}^{d_2})} \|\langle \cdot \rangle^{-s_1}\|_{L^{p'}(\mathbf{R}^{d_1})} \|\widehat{f}\|_{L_{s_1, s_2}^{p, q}(\mathbf{R}^d)}
\end{aligned}$$

which is finite when $f \in \mathcal{F}L_{s_1, s_2}^{p, q}(\mathbf{R}^d)$, $s_1 > d_1(p-1)/p$, $s_2 > d_2(q-1)/q$ for some $p, q \in (1, \infty)$. Here p' and q' denotes the conjugate exponents of p and q respectively, i.e. $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. As noticed for $p = 1$ in the previous section, different estimates are used. For $p = q = 1$ convergence follows directly from the first inequality for $f \in \mathcal{F}L_{s_1, s_2}^p$, where $s_1 \geq 0$ and $s_2 \geq 0$. In case $p = 1$ and $q \in (0, \infty)$ we have that

$$\begin{aligned}
(2\pi)^d |S^\varphi f(x, t)| &\leq \int_{\mathbf{R}^d} |\widehat{f}(\xi)| d\xi = \int_{\mathbf{R}^{d_2}} \left(\int_{\mathbf{R}^{d_1}} |\widehat{f}(\xi_1, \xi_2)| d\xi_1 \right) d\xi_2 \\
&\leq \|\langle \cdot \rangle^{-s_2}\|_{L^{q'}(\mathbf{R}^{d_2})} \left\| \int_{\mathbf{R}^{d_1}} |\widehat{f}(\xi_1, \cdot)| d\xi_1 \langle \cdot \rangle^{s_2} \right\|_{L^q(\mathbf{R}^{d_2})} \\
&= \|\langle \cdot \rangle^{-s_2}\|_{L^{q'}(\mathbf{R}^{d_2})} \|\widehat{f}\|_{L_{0, s_2}^{1, q}(\mathbf{R}^d)},
\end{aligned}$$

which is finite when $f \in \mathcal{F}L_{s_1, s_2}^{p, q}(\mathbf{R}^d)$, $s_1 \geq 0$ and $s_2 > d_2(q-1)/q$. In a similar way it follows that for $q = 1$ and $p \in (0, \infty)$, for initial data $f \in \mathcal{F}L_{s_1, s_2}^p$ no counter example concerning non-tangential convergence exist when $s_1 > d_1(p-1)/p$ and $s_2 \geq 0$. Therefore convergence along vertical lines can be extended to convergence regions when $s_1 > d_1(p-1)/p$, $s_2 > d_2(q-1)/q$ and f belongs to $\mathcal{F}L_{s_1, s_2}^{p, q}(\mathbf{R}^d)$, or for $p = 1$ and $q = 1$, $s_1 \geq 0$ and $s_2 \geq 0$, respectively.

In the proof of Theorem 4.1 we consider the function f_φ , which is defined by the formula

$$\widehat{f}_\varphi(\xi) = \widehat{f}_{\varphi, B}(\xi) = |\xi|^{-d} (\log |\xi|)^{-B} \sum_{j=1}^{\infty} \chi_j(\xi) e^{-i(x_j \cdot \xi + t_j \varphi(\xi))}, \quad (4.4)$$

where for $p, q \in (1, \infty)$, $1/p + 1/q < 1$ fixed, we may fixate $0 < B < 1$ and $0 < k < 1$ such that $1/p < Bk < 1$ and $1/q < B(1 - k) < 1$. We also have that χ_j is the characteristic function of

$$\Omega_j = \{\xi \in \mathbf{R}^d; R_j < |\xi| < R'_j\}.$$

We want to prove that f_φ in (2.1) belongs to $\mathcal{F}L_{s_1, s_2}^{p, q}(\mathbf{R}^d)$ and fulfill (4.3). The former relation is a consequence of Lemma 4.1 below, which concerns Sobolev space properties for functions of the form

$$\widehat{g}(\xi) = |\xi|^{-d}(\log |\xi|)^{-\rho/p} \sum_{j=1}^{\infty} \chi_j(\xi) b_j(\xi), \quad (4.5)$$

where χ_j is the characteristic function on disjoint sets Ω_j .

Lemma 4.1. *Assume that $0 < \rho < 1$ such that there exists $0 < k < 1$ for which $1/p < \rho k < 1$ and $1/q < \rho(1 - k) < 1$, Ω_j for $j \in \mathbf{N}$ are disjoint open subsets of $\mathbf{R}^d \setminus B_\mu(0)$, where $\mu > 2$, $b_j \in L_{loc}^1(\mathbf{R}^d)$ for $j \in \mathbf{N}$ satisfies*

$$\sup_{j \in \mathbf{N}} \|b_j\|_{L^\infty(\Omega_j)} < \infty,$$

and let χ_j be the characteristic function for Ω_j . If g is given by (4.5), then $g \in \mathcal{F}L_{s_1, s_2}^{p, q}(\mathbf{R}^d)$, where $s_1 = d_1(p - 1)/p$ and $s_2 = d_2(q - 1)/q$.

For $p = \infty$ or $q = \infty$ the result holds for $s_1 = d_1$ or $s_2 = d_2$, respectively.

Proof. Here we give the proof for $1 < p, q < \infty$. The modifications for $p = \infty$ and $q = \infty$ are left for the reader. By estimating (4.2) for the function g we get that

$$\begin{aligned} & \int_{\mathbf{R}^{d_2}} \left(\int_{\mathbf{R}^{d_1}} |\widehat{g}(\xi_1, \xi_2)|^p \langle \xi_1 \rangle^{s_1 p} d\xi_1 \right)^{q/p} \langle \xi_2 \rangle^{s_2 q} d\xi_2 \\ & \leq \int_{\mathbf{R}^{d_2}} \left(\int_{\mathbf{R}^{d_1}} |\xi|^{-dp} (\log |\xi|)^{-\rho p} \langle \xi_1 \rangle^{s_1 p} d\xi_1 \right)^{q/p} \langle \xi_2 \rangle^{s_2 q} d\xi_2, \end{aligned}$$

when $|\xi| > \mu$. Here it is sufficient to prove that the latter expression is finite. Since $|\xi|$ is bounded from below it follows that $|\xi_1|$ and $|\xi_2|$ can not be arbitrarily small at the same time. The integral is divided into

the three following parts

$$\begin{aligned} & \int_{|\xi_2|>\mu} \left(\int_{|\xi_1|<\mu} |\xi|^{-dp} (\log |\xi|)^{-\rho p} \langle \xi_1 \rangle^{s_1 p} d\xi_1 \right)^{q/p} \langle \xi_2 \rangle^{s_2 q} d\xi_2 \\ & \leq \int_{|\xi_2|>\mu} |\xi_2|^{-dq} (\log |\xi_2|)^{-\rho q} \left(\int_{|\xi_1|<\mu} \langle \xi_1 \rangle^{s_1 p} d\xi_1 \right)^{q/p} \langle \xi_2 \rangle^{s_2 q} d\xi_2, \end{aligned}$$

which is finite since $q(d_1 - d) < 0$,

$$\begin{aligned} & \int_{|\xi_2|<\mu} \left(\int_{|\xi_1|>\mu} |\xi|^{-dp} (\log |\xi|)^{-\rho p} \langle \xi_1 \rangle^{s_1 p} d\xi_1 \right)^{q/p} \langle \xi_2 \rangle^{s_2 q} d\xi_2 \\ & \leq \int_{|\xi_2|<\mu} \left(\int_{|\xi_1|>\mu} |\xi_1|^{-dp} (\log |\xi_1|)^{-\rho p} \langle \xi_1 \rangle^{s_1 p} d\xi_1 \right)^{q/p} \langle \xi_2 \rangle^{s_2 q} d\xi_2, \end{aligned}$$

which is finite since $p(d - d_2) < 0$ and

$$\begin{aligned} & C \left(\int_{|\xi_1|>\mu} |\xi_1|^{-d_1 p} (\log |\xi_1|)^{-\rho k p} \langle \xi_1 \rangle^{s_1 p} d\xi_1 \right)^{q/p} \\ & \quad \cdot \left(\int_{|\xi_2|>\mu} |\xi_1|^{-d_2 q} (\log |\xi_2|)^{-\rho(1-k)q} \langle \xi_2 \rangle^{s_2 q} d\xi_2 \right) \\ & \leq 2^{(s_1 p + s_2 q)/2} C \left(\int_{\rho}^{\infty} \frac{1}{r(\log r)^{\rho k p}} \right) \left(\int_{\rho}^{\infty} \frac{1}{r(\log r)^{\rho(1-k)q}} \right) dr < \infty. \end{aligned}$$

The second inequality holds since $(1 + r^2)^{sp/2} < (r^2 + r^2)^{sp/2} = 2^{sp/2} r^{sp}$ for $r > 1$. The last inequality follows by choosing $0 < k < 1$ such that $\rho k p > 1$ and $\rho(1 - k)q > 1$. \square

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